Dynamic separation of chaotic signals in the presence of noise

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The problem of separation of a noise-contaminated observed sum of chaotic signals into the individual components is considered. A noise threshold is found above which high-quality separation is impossible. Below the threshold, each signal is recovered with any prescribed accuracy with a separation method. The threshold effect is shown to be associated with the information content of chaotic signals and a theoretical estimate is given for the threshold.

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I. INTRODUCTION

Chaotic oscillators exist in many physical, biological, electronic, mechanical, and other systems. Oscillations produced by such sources are often analyzed with the use of observations of a single component of the process. Besides, the observed signal can represent not the pure component of the process, but its certain transformation or it can be corrupted by uncontrollable distortions. Typical problems that an observer confronts by analyzing chaotic sources are cleaning chaotic signals off noise [1–6], reconstructing chaotic attractor [7–9], and estimating its correlation dimension [10–12].

The situation becomes more complicated in the case of two or more chaotic sources, when the observer receives a certain combination of their signals, e.g., the sum, in the simplest case. In order to analyze each source, the observer has to separate the signals from the observed sum into the individual components. Can this problem be solved in the case of a single observable?

As was shown in Ref. [13], this was possible, in principle, in the case that the observer knew the equations describing the dynamic systems. The authors used the principle of chaotic synchronization to demonstrate this possibility. Unfortunately, the method using chaotic synchronization proved to be very sensitive to external noise.

In this paper we use a different approach based on iteration of chaotic systems in reverse time. We discuss this approach on the basis of an example of two chaotic sources represented by two maps of logistic parabola with different parameters. On the basis of this simple example we show that chaotic signals can be separated not only in the absence but also in the presence of noise. As is found, there is a certain threshold of the noise value above which high-grade separation becomes impossible. Below the threshold, each signal can be recovered with any prescribed (high) accuracy. We discuss this effect and determine that it is coupled not with the concrete method of separation, but with basic reasons associated with the information content of the chaotic signals and with impossibility for an observer to receive this information content without serious distortions in the presence of noise stronger than a certain threshold.

We also give a theoretical estimate for the threshold noise value and compare the threshold obtained numerically with the theoretical estimate. The difference between the two threshold values is a measure of the concrete algorithm efficiency. The less the difference, the more efficient is the algorithm. We introduce a "multibranch" algorithm for chaotic signal separation and demonstrate that it can provide separation efficiency close to the theoretical estimate.

Finally, we discuss to what extent the obtained results can be generalized to other chaotic sources.

II. SEPARATION METHOD

The problem we consider here is separation of chaotic signals in the case when observer knows exact equations describing the chaotic sources. Let there be two oscillators producing chaotic signals $x_j(k)$, $j=1,2$; and k be discrete time. On the path to observer the signals $x_j(k)$ are summed. In general, the sum signal is also contaminated by an additive noise $\xi(k)$ (Fig. 1). The observer has to separate the individual signals from the sum.

The oscillator dynamics equations are

$$x_1(k+1)=f_1(x_1(k)),$$
$$x_2(k+1)=f_2(x_2(k)). \tag{1}$$

The observed signal is

$$u(k)=x_1(k)+x_2(k)+\xi(k). \tag{2}$$

So, the problem can be rigorously defined as follows. Given a sequence of sum signal values $\{u(k)\}$, $k=1,2,\ldots,N$; knowing the dynamics of the systems generating the chaotic signals (here, the functions f_1 and f_2), and given (good) estimates $\tilde{x}_1(N)$ and $\tilde{x}_2(N)$ at Nth time moment; to obtain estimate sequences $\tilde{x}_1(k)$ and $\tilde{x}_2(k)$, k

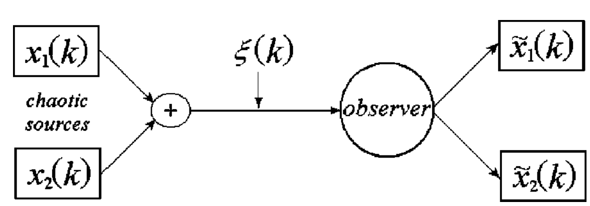

FIG. 1. Separation of chaotic signals.

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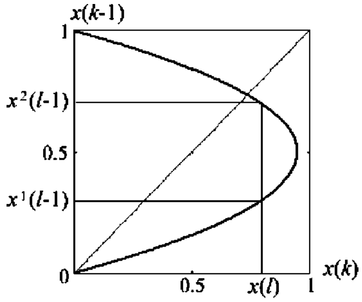

FIG. 2. Two-valued function $f^{-1}(x)$ of the map inverse to logistic map.

$=1,2,\ldots,N$, of the oscillator signals $x_1(k)$ and $x_2(k)$ on the entire time interval. The estimate sequences must satisfy the dynamics of sources (1) and must be the most close to $x_1(k)$ and $x_2(k)$, respectively.

Let for certainty the chaotic sources be described by maps of logistic parabola $f(x)=\mu x(1-x)$

$$x_1(k+1)=\mu_1 x_1(k)[1-x_1(k)],$$

$$x_2(k+1)=\mu_2 x_2(k)[1-x_2(k)]. \tag{3}$$

The idea of the proposed separation method is as follows. The observer has the maps f^{-1} inverse to those that generate the chaotic signals (Fig. 1):

$$x_1(k-1)=f_1^{-1}(x_1(k)),$$

$$x_2(k-1)=f_2^{-1}(x_2(k)). \tag{4}$$

Iteration of maps (4) is equivalent to backward iteration of Eqs. (1). Let the observer at kth moment have separate estimates of the values of the chaotic signals of both oscillators, i.e., an estimate $\tilde{x}_1(k)$ for $x_1(k)$ and $\tilde{x}_2(k)$ for $x_2(k)$. Iteration of the maps of inverse systems (4) with initial conditions $\tilde{x}_1(k)$ and $\tilde{x}_2(k)$ gives estimates of the signals at $(k-1)$th moment (Fig. 2).

Since maps (3) are stretching on the average (over the attractors), inverse maps (4) are contracting (on the average). Hence, the estimates for signals x_1 and x_2 at $(k-1)$th moment, obtained from the estimates $\tilde{x}_1(k)$ and $\tilde{x}_2(k)$, will on the average be more accurate that the initial estimates $\tilde{x}_1(k)$ and $\tilde{x}_2(k)$. However, maps (4) are two valued (Fig. 2), i.e., map function has two "branches," and each iteration of Eq. (4) gives two values for a single argument: two potential estimates $\tilde{x}_1^1(k-1)$ and $\tilde{x}_1^2(k-1)$ for the first source, and two estimates $\tilde{x}_2^1(k-1)$ and $\tilde{x}_2^2(k-1)$ for the second source. So, we have to choose the "proper" branch of each map function by iteration. This can be organized as follows. These two pairs of two estimates give us four possible combinations for the sum signal estimates at $(k-1)$th moment:

$$u_{ij}(k-1)=\tilde{x}_1^i(k-1)+\tilde{x}_2^j(k-1), \quad i,j=1,2. \tag{5}$$

At the same time, we know that the observed signal value at $(k-1)$th moment was $u(k-1)$. We can make the proper choice if we compare the value of $u(k-1)$ with those of $u_{ij}(k-1)$. Indeed, the best choice (i,j) is the combination of the branches that minimizes deviation of the sum of estimates from the observed sum signal at that moment:

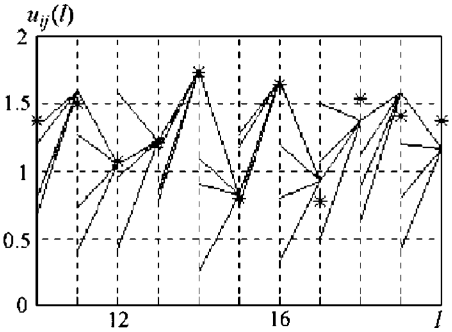

FIG. 3. Choice between estimate combinations. l is discrete time. The values of the observed sum signal $u(l)$ are denoted by asterisks.

$$(i,j):|u(k-1)-u_{ij}(k-1)| = \min_{p,q}|u(k-1)-u_{pq}(k-1)|, \quad p,q=1,2. \tag{6}$$

The scheme for the choice of proper branch combinations is illustrated in Fig. 3. The values of the observed signal $u(l), l<k$, are denoted by asterisks. At lth moment from four possible values of $u_{ij}(l)$ we take the one most close to $u(l)$, the same is done at $(l-1)$th moment, and so on. Thus, successive application of the discussed procedure allows us to separate the signals on the entire time interval $(1,N)$.

If λ is Lyapunov exponent of a map (averaged over the map attractor), then the average stretching factor of the map is e^{λ}, and the inverse map contraction factor is $e^{-\lambda}$. So, the estimate errors $\delta_1(l)$ and $\delta_2(l)$ of the separated signals $\tilde{x}_1(l)$ and $\tilde{x}_2(l)$ decrease exponentially (on the average)

$$\delta_1(l)=|\tilde{x}_1(l)-x_1(l)|=\delta_1(N)\exp[-\lambda_1(N-l)],$$

$$\delta_2(l)=|\tilde{x}_2(l)-x_2(l)|=\delta_2(N)\exp[-\lambda_2(N-l)], \tag{7}$$

where $\delta_1(N)=|\tilde{x}_1(N)-x_1(N)|$ and $\delta_2(N)=|\tilde{x}_2(N)-x_2(N)|$ are initial estimate errors, and λ_1 and λ_2 are Lyapunov exponents of the trajectories of the first and the second systems, respectively.

In agreement with expression (7), the closeness of the signals $\tilde{x}_1$ and $\tilde{x}_2$ recovered by the observer to the signals x_1 and x_2 of the sources improves exponentially with each step of inverse function iteration and eventually achieves the limit of calculation accuracy. In numerical experiments with accuracy ε the limit attainable closeness, i.e., separation accuracy, is achieved after

$$T_{conv}=-\ln(\varepsilon/\delta)/\lambda \tag{8}$$

steps at most, where δ is the initial estimate error. Thus, the described procedure allows one to separate the signals, given on time interval $(1,N)$, nearly on the entire interval $(1,N$

$-T_{conv}$) as accurately as the machine arithmetic, with a few less accurate samples at the end of the separated signal sequences. This less accuracy of the ending T_{conv} samples can be explained by the lack of information necessary to separate the signals on the interval $(N-T_{conv},N)$.

Above, we discussed the method for signal separation under condition that estimates $\tilde{x}_1(N)$ and $\tilde{x}_2(N)$ of the chaotic sources' signals are known at Nth moment. If the estimates are good, e.g., the initial error is of the order of the machine accuracy $\delta_{1,2}(N)\approx\varepsilon$, then $T_{conv}=0$, and the signals are separated as accurately as possible on the entire time interval $(1,N)$. However, in general, one may take any pair of points $\tilde{x}_1(N)$ and $\tilde{x}_2(N)$ on the attractors of maps (3) as the initial estimates. Having started from these initial conditions the calculated trajectories of systems (4) converge with time to the trajectories of the chaotic oscillators. It is the time of convergence T_{conv} that depends on the particular choice of the initial points.

III. VERIFICATION OF THE METHOD AND EFFICIENCY MEASURES

The efficiency of the proposed method for chaotic signal separation was investigated on example of chaotic sources described by logistic maps (3) with the parameters set at $\mu_1=3.7$ and $\mu_2=3.8$ (Lyapunov exponents $\lambda_1=+0.355$ and $\lambda_2=+0.432$, respectively).

In the absence of noise, the signals of the two logistic maps are efficiently separated. Computer simulation with double precision arithmetic provides the limit attainable accuracy equal to $\varepsilon\approx10^{-16}$ (16 significant digits). The maximum value of the time of convergence T_{conv} of the estimate trajectories to the real, source trajectories corresponds to the worst initial estimates $\delta_{1,2}(N)=1$, i.e., $T_{conv}=-\ln(\varepsilon)/\lambda$, where $\lambda=\min(\lambda_1,\lambda_2)$. Here, for $\lambda=0.355$ $T_{conv}=-\ln(10^{-16})/\lambda=109$ steps, while rather good practical accuracy of 10^{-3} is achieved already after 20 steps.

The estimate of maximum convergence time T_{conv} obtained from expression (8) is exact in the case of the maps whose stretching factor $L=e^{\lambda}$ is constant in every point of the attractor, such as Bernoulli shift or symmetrical tent maps. In the case of logistic map, however, the stretching coefficient varies over a wide range; there are stretching and contracting regions in its phase space, so expression (8) gives some average value. In order to analyze the convergence time T_{conv}, we calculated a distribution of T_{conv} for 40 000 initial estimates taken at random over the corresponding attractors of maps (3) (Fig. 4). As can be concluded from the plot of T_{conv}, the right boundary of the distribution practically coincides with the maximum estimate for T_{conv} from Eq. (8). Nearly half the initial estimates give the maximum T_{conv}, and to the left the distribution rapidly decreases. Gaps in the distribution can be explained by the presence of contracting regions in logistic map.

Since the method for chaotic signal separation proved to be applicable in the absence of noise, we concentrated our further investigation on the method resistance with respect to external noise. Gaussian, normally distributed noise $\xi(k)$ with variance σ was added to the sources' signals. The signal recovery error $x_i(k)-\tilde{x}_i(k), i=1,2$, was treated as residual noise in the separated signals. The signal-to-noise ratio (SNR) of the separated signals η_{out} was calculated as a function of the sum signal SNR η_{in},

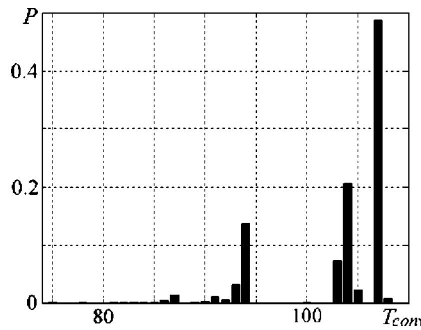

FIG. 4. Distribution of convergence time T_{conv}.

$$\eta_{out}=\langle x_i^2\rangle/\langle(x_i-\widetilde{x_i})^2\rangle,\quad i,j=1,2.$$

$$\eta_{in}=\langle(x_1+x_2)^2\rangle/\sigma^2,$$

$$\eta_{in}=10\log_{10}[\langle(x_1+x_2)^2\rangle/\sigma^2]\ \ \mathrm{dB} \tag{9}$$

(all signals were normalized to zero mean values).

In order to quantitatively estimate performance of the separation method, we need a measure of separation efficiency. That is, in what case the signals can be considered separated, provided that the separated signals do not coincide with those of the sources? We considered several efficiency measures. The first is the difference between the noise levels of the separated signals and of the observed signal $u(k)$. Separation is considered effective when the noise level in the separated signals is lower than the input noise level, and ineffective in the opposite case. We consider the threshold value of the external noise that gives $\eta_{out}=\eta_{in}$ as the boundary of separation ability with respect to the additive noise.

Calculation results are presented in Fig. 5. The results of the above method are represented by curve *A*, which shows that the region of effective separation extends to $\eta_{in}\approx65$ dB. Given for comparison is curve *C*, corresponding to results for the method of chaotic synchronization [13]. As can be seen in Fig. 5(a), the noise in the signals separated by the method of chaotic synchronization is always higher than the input noise level and condition $\eta_{out}>\eta_{in}$ is never satisfied. Note that to the right of the separation boundary ($\eta_{in}\approx65$ dB) the noise in the separated signals rapidly decreases according to relations (7), and its value is eventually determined by only the number of backward iteration steps and by the machine calculation accuracy. This means that the signals are not only separated but also cleaned off noise.

To the left of the separation boundary (in the region of $\eta_{in}<65$ dB) in the process of separation, sporadic separation error bursts can occur (Fig. 6). Even a single and very short burst can seriously spoil the separation characteristics. However, if the additive noise is relatively small, the bursts occur seldom and most of the time the chaotic signals are well separated. Therefore, we also used another efficiency

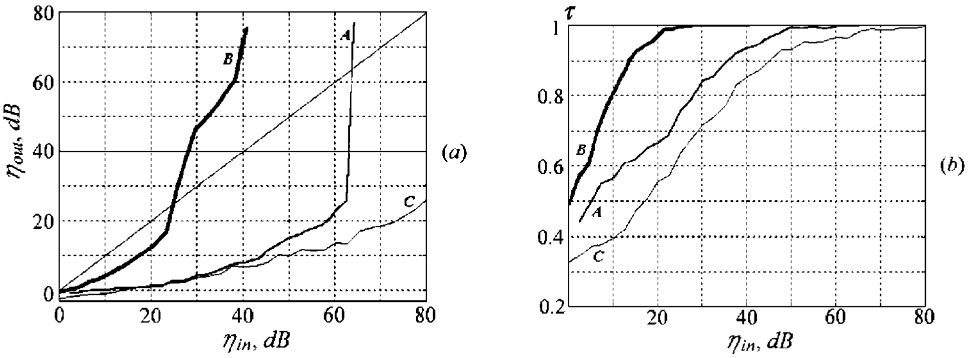

FIG. 5. (a) Signal-to-noise ratio of the separated signals, η_{out}, and (b) relative separation time τ as functions of sum signal SNR η_{in}. Curves are shown for (*A*) single-branch algorithm, (*B*) algorithm with 16 branches, and (*C*) method of Ref. [13]. The results were averaged over 10 000-sample chaotic sequences.

measure, the relative time τ of effective signal separation, which is defined as a fraction of the total time within which the local signal estimation error $\delta_{1,2}=|x_{1,2}(k)-\tilde{x}_{1,2}(k)|$ is less than $\delta_{1,2}<0.01$ [Fig. 5(b)]. Evidently, separation is effective if τ is close to one. Here, the boundary is $\eta_{in}\approx 60$ dB. As can be seen in Figs. 5(a) and 5(b), both measures give close values of the separation boundaries.

Another efficiency measure may be the value of the added noise level at which the method provides a certain prescribed quality of separation. We assigned the required quality to rms (root-mean-square) signal error of the order of 0.01, since it is quite a good recovery accuracy. The above method gives rms error $=0.01$ at $\eta_{in}=62$ dB.

Simulation of the separation procedure shows that with increasing external noise the rate of the error bursts also increases, which gradually ruins the method efficiency. Analysis of the recovered signal waveforms shows that strong external noise $\xi(k)$ at a particular moment can considerably shift the actual sum of the sources' signals (2) and result in error bursts due to a wrong choice among the inverse map branches at that moment [see relation (6) and Fig. 3]. This wrong choice is exhibited on the next step of (4) map iteration as a sharp burst of separation error. Then the separated trajectories again begin to converge to the sources' trajectories, which can take a number of steps. These irregular "error" bursts are the reason for the residual noise.

IV. THRESHOLD EFFECT AND ITS NATURE

As can be seen in Fig. 5, at the boundary value of the external noise ($\eta_{in}\approx 65$ dB), the separated signal noise level jumps by more than 40 dB. Is the presence of such a threshold property of the discussed separation method, or a common feature of chaotic signal separation? And a more general question: are there principle limitations on separation

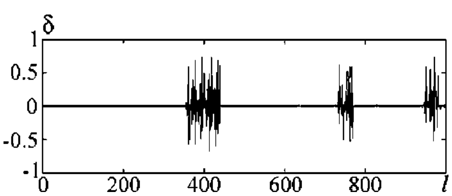

FIG. 6. Bursts of signal separation error $\delta=x_1-\tilde{x}_1$; $\eta_{in}\approx 40$ dB; *l* is discrete time.

of chaotic signals and what are the reasons for these limitations?

To answer these questions, let us consider information properties of chaotic signals. A one-dimensional (1D) chaotic map generates Kolmogorov entropy with a mean rate equal to Lyapunov exponent λ [8]. In the discussed case, Kolmogorov entropy is equivalent to information *I*, which is used, however, to be expressed in bits per iteration. Thus, chaotic 1D map generates

$$I=\lambda/\ln 2 \tag{10}$$

information bits per iteration. For example, logistic maps with parameters $\mu_1=3.7$ and $\mu_2=3.8$ have Lyapunov exponents $\lambda_1=0.355$ and $\lambda_2=0.432$ and generate on the average $I_1=0.51$ and $I_2=0.62$ bits per iteration, respectively. These are mean values, however, the amount of generated information differs from iteration to iteration [Figs. 7(a), 7(b)].

According to expression (2) observer receives the sum of chaotic signals x_1 and x_2 distorted by noise ξ. On each iteration step the sum of signals contains certain amount of information whose distribution density is presented in Fig. 7(c). In order to separate the signals x_1 and x_2, it is necessary that the information is not lost due to contamination of the signal sum by the noise ξ. Note that one can treat expression (2) as a model of a "communication channel" with Gaussian noise through which a signal $x(k)=x_1(k)+x_2(k)$ is transmitted. According to Shannon theorem [14], the information-carrying capacity of the channel per iteration is equal to

$$C=\frac{1}{2}\log_2\left(1+\frac{\langle x^2(k)\rangle}{\sigma^2}\right)=\frac{1}{2}\log_2(1+\eta_{in}). \tag{11}$$

Maximum amount of information going through this noisy channel is determined by the right boundary of the distribution density I_{max} in Fig. 7(c). This gives a necessary condition for the signal separation

$$C>I_{max}, \tag{12}$$

consequently,

$$\eta_{in}>2^{2I_{max}}-1. \tag{13}$$

In the discussed case, $I_{max}\approx 3.4$ bits per iteration, hence

$$\eta_{in}>20 \text{ dB}. \tag{14}$$

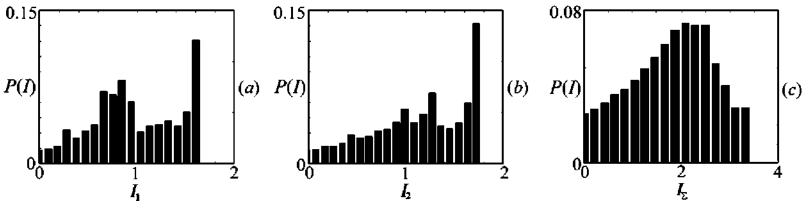

FIG. 7. Per-iteration distribution density of information produced by logistic map with the parameter set at (a)$\mu_1 = 3.7$; at (b) $\mu_2 = 3.8$; and (c) of the sum of the two signals.

Comparison of the obtained estimate (14) with curve A in Fig. 5 indicates that the difference between the theoretical and numerically obtained values of the separation boundary is greater than 40 dB.

V. MULTIBRANCH METHOD

In the algorithm that was used above, the decision on which branch to take was made locally, in one point of time domain, and the preceding and the following histories were not taken into account. So, we developed and investigated a modified method whose efficiency is improved due to the use of nonlocal information on each iteration. We backtrack several branches simultaneously besides the one, optimal in the sense of condition (6), and choose among them by means of minimizing the deviation signal averaged over a certain time interval.

To do this, we build a tree of possible trajectories of $\tilde{x}_1$ and $\tilde{x}_2$ on the given interval (l,k) and take the pair that minimizes the functional G on this interval (l,k)

$$G = \sum_{i=j}^{k} \{u(i) - [\tilde{x}_1(i) - \tilde{x}_2(i)]\}^2. \tag{15}$$

Due to evident inevitable restrictions on computational capabilities, memory resources, etc., we restrict the number of the backtracked branches, say to M, "best" in a certain sense, by means of discarding the least probable ones. Besides, specific dynamics of chaotic systems is to be taken into account: since the backtracked branches tend to converge due to relation (7), from time to time we remove the "stuck" ones in order to keep branches different. When the entire interval $(1,N)$ is processed, the separated signals $\tilde{x}_1$ and $\tilde{x}_2$ are obtained with the condition of the minimum G.

The results of separating chaotic signals with this algorithm are presented in Figs. 5(a) and 5(b) (curve B). Sixteen branches were tracked back ($M = 16$). The results indicate that with this algorithm the boundary of effective separation is shifted toward $\eta_{in} = 25-30$ dB, which is $35-40$ dB better than with the algorithm with single branch and much closer to the theoretical separation limit of 20 dB.

VI. CONCLUSIONS

In this paper we considered the problem of separation of an observed sum of chaotic signals into the individual components. For the case of exactly known equations of the chaotic sources we proposed a separation method based on backward iteration of the source equations. As was shown, with this method chaotic signals can be efficiently separated. If the observed sum of chaotic signals is contaminated by additive noise, the method is also functional. A certain noise threshold is found above which good-quality separation is impossible, while below the threshold, each signal can be recovered with precision limited by only the machine arithmetic. Investigation of this threshold effect have shown that it is associated with the information content of chaotic signals, and with the help of the information theory a theoretical estimate for the noise threshold is obtained, which is the limit value for separation of chaotic signals in the presence of noise.

By numerical simulation using developed algorithms we demonstrated separation of the signals of two logistic maps at the sum-signal–to–noise ratio as low as 25–30 dB, while the theoretical estimate for the separation boundary was about 20 dB.

Though we discussed here the problem of separation of chaotic signals in the case of one-dimensional systems, the above estimation of the separation limits is valid for a broader class of chaotic sources.

The proposed separation methods can be directly generalized to the case of $m > 2$ chaotic sources (represented by 1D systems), and also to the sources described by multidimensional hyperbolic maps that have both stable and unstable manifolds (the results will be given elsewhere). A possibility of generalization of the method to other systems needs further investigations.

ACKNOWLEDGMENT

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